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STABILITY OF TWO-DIMENSIONAL TRAVELING WAVES ON A VERTICALLY DRAINING LIQUID FILM TO THREE-DIMENSIONAL PERTURBATIONS

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UDC 532.51

It is known that for practically all Reynolds numbers waves exist on the surface of a liquid layer draining along a vertical tube. This is because the flow of a film with a smooth free surface is unstable [1]. Unless special measures are taken (such as the creation of uniform conditions around the perimeter of the tube at the inlet) the waves are three-dimensional and irregular [2] and are extremely sensitive to external perturbations. Hence the theoretical or experimental study of a draining film is very difficult.

The wave flow can be regularized by applying pulsations to the flow rate or by creating uniform conditions at the flow inlet [2, 3]. Then there exists a region of two-dimensional (annular) regular waves whose length depends strongly on the properties of the liquid and the flow rate [3], and which evolves into three-dimensional flow [3].

By superimposing pulsations of different frequencies, two-dimensional waves of different lengths can be generated. There exist two types of waves with very different properties: quasiharmonic and solitary waves [2, 3]. The system of equations of [6] for the instantaneous thickness and flow rate of the liquid was used in [4, 5] to calculate different two-dimensional nonlinear steady traveling waves. Some of these wave processes were found to be in good quantitative agreement with experiment. The stability of these wave processes to plane infinitesimal perturbations was studied in [7-9] by means of bifurcation analysis. Two types of waves were preferred in the sense of stability. These preferred waves will be referred to as belonging to the first and second families, and they correspond to the quasiharmonic and solitary waves observed experimentally.

The equations of [6] were extended to the case of three-dimensional perturbations in [10] by averaging the equations of motion in the direction perpendicular to the layer (the y direction) and assuming certain velocity profiles in the x direction (along the gravitational acceleration vector) and in the z direction. These equations can be written in the form

$$\frac{\partial q}{\partial t}+1{,}2\left(\frac{\partial}{\partial x}\frac{q^2}{h}+\frac{\partial}{\partial z}\frac{qQ}{h}\right)=-\frac{3\nu q}{h^2}+gh+\frac{\sigma h}{\rho}\left(\frac{\partial^3 h}{\partial x^3}+\frac{\partial^3 h}{\partial x\,\partial z^2}\right),$$

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$$\frac{\partial Q}{\partial t}+1{,}2\left(\frac{\partial}{\partial x}\frac{qQ}{h}+\frac{\partial}{\partial z}\frac{Q^2}{h}\right)=-\frac{3\nu Q}{h^2}+\frac{\sigma h}{\rho}\left(\frac{\partial^3 h}{\partial z^3}+\frac{\partial^3 h}{\partial z\,\partial x^2}\right),$$
$$\frac{\partial h}{\partial t}+\frac{\partial q}{\partial x}+\frac{\partial Q}{\partial z}=0. \tag{1}$$

Here h is the instantaneous thickness of the film; q is the instantaneous flow rate in the film in the x direction; Q is the instantaneous flow rate in the z direction; ν is the kinematic viscosity; σ is the surface tension; g is the acceleration due to gravity; ρ is the density of the liquid.

In the present paper we consider the stability of two-dimensional periodic traveling-wave solutions of (1) to three-dimensional perturbations. The stability of traveling two-dimensional waves on a film to plane perturbations was first considered in [7]. The stability of waves of the first family was studied in the neighborhood of the neutral stability line and it was found that they are unstable to plane perturbations for moderately large values of Re. Numerical methods were used in [8, 9] to study the stability of waves of the second family and also waves of the first family to plane perturbations for wider intervals of wave number and Reynolds number than assumed in [7]. The boundaries of the stable bands were found for waves of both families for moderate Re. The main innovation of the present paper is the study of the stability of two-dimensional nonlinear waves to three-dimensional perturbations in the region where these waves are stable to plane perturbations.

The two-dimensional periodic waves calculated in [5, 8] are denoted here as $h_0(\xi)$, $q_0(\xi)$ ($\xi = x - ct$, c is the phase velocity of the wave). Substituting $h(x, z, t) = h_0(\xi) + h'(\xi, z, t)$, $q(x, z, t) = q_0(\xi) + q'(\xi, z, t)$, $Q(x, z, t) = Q'(\xi, z, t)$ into (1) and linearizing, we obtain the following equations determining the stability of the solution (h_0, q_0, 0):

$$\frac{\partial q'}{\partial t}+A\frac{\partial q'}{\partial \xi}+Bq'+K\frac{\partial h'}{\partial \xi}+Dh'-3h_0\frac{\partial^3 h'}{\partial \xi^3}+1{,}2\frac{q_0}{h_0}\frac{\partial Q'}{\partial z}-3h_0\frac{\partial^3 h'}{\partial \xi\,\partial z^2}=0,$$
$$\frac{\partial h'}{\partial t}-c\frac{\partial h'}{\partial \xi}+\frac{\partial q'}{\partial \xi}+\frac{\partial Q'}{\partial z}=0, \tag{2}$$
$$\frac{\partial Q'}{\partial t}+A_1\frac{\partial Q'}{\partial \xi}+B_1Q'-3h_0\frac{\partial^3 h'}{\partial z^3}-3h_0\frac{\partial^3 h'}{\partial z\,\partial \xi^2}=0,$$

where

$$A=2{,}4\frac{q_0}{h_0}-c;\quad B=2{,}4\frac{d}{d\xi}\left(\frac{q_0}{h_0}\right)+\frac{p}{h_0^2};$$
$$A_1=1{,}2\frac{q_0}{h_0}-c;\quad B_1=1{,}2\frac{d}{d\xi}\left(\frac{q_0}{h_0}\right)+\frac{p}{h_0^2};$$
$$K=-1{,}2\frac{q_0^2}{h_0^2};\quad D=-\left(1{,}2\frac{d}{d\xi}\left(\frac{q_0^2}{h_0^2}\right)+2p\frac{q_0}{h_0^3}+F+3\frac{d^3h_0}{d\xi^3}\right).$$

In (2) the variables have been made dimensionless in accordance with [8]. The basic parameter is $p=(27\mathrm{Fi}/\mathrm{Fr}\,\mathrm{Re}^{10})^{1/6}$ [$\mathrm{Fi}=(\sigma/\rho)^3/g\nu^4$, $\mathrm{Fr}=\langle q_0\rangle^2/g\langle h_0\rangle^3$, $\mathrm{Re}=\langle q_0\rangle/\nu$ and the angular brackets imply an average over the wavelength]. The dimensionless number F and the phase velocity c were calculated in [5, 7].

Since the variables (t, z) do not appear explicitly in (2), its solution can be written in the form

$$(h', q', Q')=(h_1, q_1, Q_1)e^{-\gamma t+i\beta z}+\text{c.c.} \tag{3}$$

(c.c. denotes the complex conjugate and β is a real parameter).

Substitution of (3) into (2) leads to a system of ordinary differential equations with periodic coefficients in ξ:

$$\hat{L}(q_1, h_1, Q_1)=\gamma(q_1, h_1, Q_1) \tag{4}$$

($\hat{L}$ is a matrix differential operator). We consider the stability to bounded perturbations in the coordinate ξ. It follows from Floquet's theorem that the solutions of (4) have the form

$$(q_1, h_1, Q_1)=e^{i\alpha L\xi}\,(\psi(\xi), \varphi(\xi), \chi(\xi)). \tag{5}$$

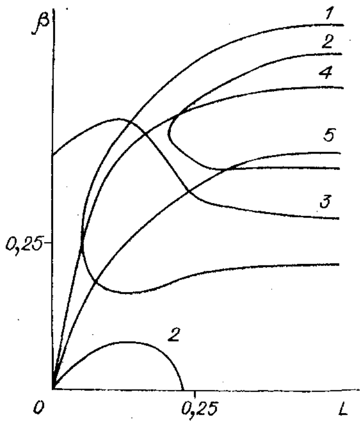

Fig. 1

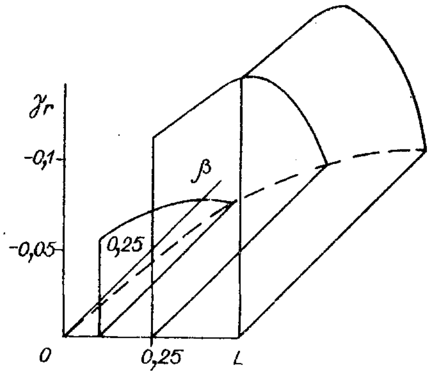

Fig. 2

Here (ψ, φ, χ) are periodic functions with the same period $\lambda = 2\pi/\alpha$ as the solution (h_0, q_0) being studied for stability; L is a real parameter between zero and one.

After substituting (5) into (4), the stability problem reduces to finding γ such that the resulting system of equations has a periodic solution. The numerical algorithms are similar to those discussed in [8]. The wave (h_0, q_0, 0) is stable if the real parts of all γ are positive [$\mathrm{Real}(\gamma) \geqslant 0$] for all values of β and L.

The stability problem for plane perturbations ($\beta = 0$) was studied in [8] and it was found that waves of the first family (which correspond to plane-parallel flow when $\alpha = 1$) are stable against plane perturbations only in a narrow band of wave numbers (and then only for $p \geq 4$ (small Re)).

The results of the present paper show that waves of the first family are unstable against three-dimensional perturbations in the entire region of wave number where these waves exist and for all values of p.

Curves 1-5 of Fig. 1 bound the regions of amplified three-dimensional perturbations for waves with α = 0.79, 0.75, 0.6, 0.74, 0.55 (p = 10 for curves 1-3 and p = 1 for curves 4 and 5). Perturbations with $\beta \gtrsim 0.7$ are damped. The region of instability is practically symmetric with respect to the line L = 0.5 and so it is not shown completely in Fig. 1.

The calculations also show that the waves are unstable against perturbations of a different kind, depending on the wave number. This is seen in Fig. 1, where there are two unstable regions for $\alpha = 0.75$ (curve 2). A similar result was obtained in [11], where the stability of waves of the first family to three-dimensional perturbations was studied using equations holding for Re $\lesssim 1$ (large p). The results of our calculations with $p \gtrsim 100$ agree quantitatively with the results of [11], which is evidence in favor of the assumed equations (1).

In contrast to the results of [11], our calculations show that only the real parts of the eigenvalues vanish on the boundaries of the stable bands, except in certain cases at the special points L = 0.5 and 0, where the imaginary parts of the eigenvalues may also vanish. In Fig. 1 these special points are L = 0.5 for curves 1, 2, 4 and L = 0 for curve 3. Three-dimensionally periodic steady traveling waves branch off from these points.

The growth factor γ_r of the amplified perturbations are shown in Figs. 2 and 3 for p = 1 for waves with α = 0.7 and 0.55, respectively. Perturbations with L = 0.5 are the most critical.

Next, we consider the stability of waves of the second family. It was shown in [8] that waves of the second family (unlike waves of the first family) have a set of bands which are stable against plane perturbations. It is shown in the present paper that waves of this family are unstable against three-dimensional perturbations for the entire wave-number and Reynolds number region in which the waves exist.

In Fig. 4 the curves 1-3 bound the regions of amplified perturbations for the three waves α = 0.52, 0.48, 0.46 for p = 10. The value $\alpha = 0.52$ practically corresponds to the upper boundary of the existence region of waves of the second family. Calculation of the

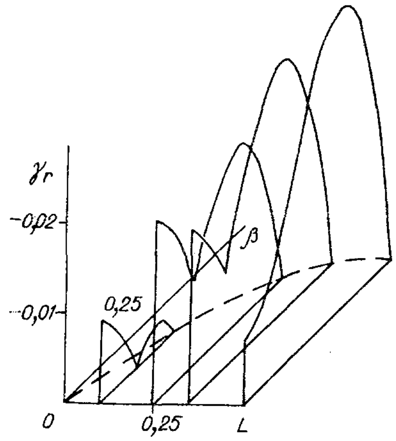

Fig. 3

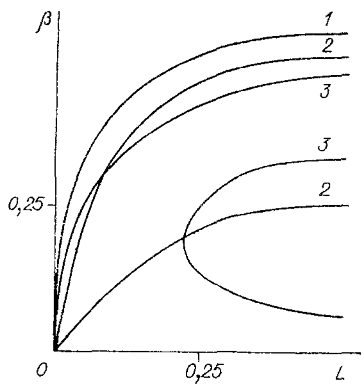

Fig. 4

stability of waves with $\alpha \lesssim 0.4$ is quite difficult because the initial wave is composed of a large number of harmonics. Therefore analytical methods were used to study the stability against long-wavelength modulations (small β and L).

Introducing the small parameter ε ($\varepsilon = \sqrt{\beta^2 + \alpha^2 L^2}$, $\beta = \varepsilon \sin\theta$, $\alpha L = \varepsilon \cos\theta$, $0 \leqslant \theta \leqslant \pi/2$) and writing the solution of the system (4) and (5) in the series form

$$(\psi, \varphi, \chi) = \sum_{n=0}^{3} (\psi_n, \varphi_n, \chi_n)\, \varepsilon^n; \quad \gamma = \sum_{n=0}^{3} \gamma_n \varepsilon^n,$$

to zero order in ε we obtain the equation

$$\widehat{L}_0(\psi_0, \varphi_0, \chi_0) = \gamma_0(\psi_0, \varphi_0, \chi_0) \tag{6}$$

($\hat{L}_0$ is a matrix differential operator). The solution of (6) is written as

$$(\psi_0, \varphi_0, \chi_0) = \left(\frac{dq_0}{d\xi}, \frac{dh_0}{d\xi}, 0\right), \quad \gamma_0 = 0.$$

To order ε^1:

$$\widehat{L}_0(\psi_1, \varphi_1, \chi_1) = \gamma_1(\psi_0, \varphi_0, \chi_0) + i\cos\theta\,(f_1, f_2, 0) + i\sin\theta(0, 0, f_3) \tag{7}$$

(f_1, f_2, f_3 are known periodic functions). The system of equations (7) has a solution if the right-hand side is orthogonal to the solutions of the homogeneous form of the adjoint problem to (6). One of the nontrivial solutions of this problem has the form

$$(\psi^*, \varphi^*, \chi^*) = (0, 1, 0). \tag{8}$$

It was verified numerically that there are no other nontrivial solutions of the adjoint problem at nonsingular points.

All three terms on the right-hand side of (7) are orthogonal to (8) and its solution can be written in the form

$$(\psi_1, \varphi_1, \chi_1) = -\gamma_1(\alpha_1, \alpha_2, 0) + i\cos\theta(\beta_1, \beta_2, 0) + i\sin\theta(0, 0, r), \tag{9}$$

where the real periodic functions α_1, α_2, β_1, β_2, r are found numerically. The existence condition for a solution in the next approximation in ε gives a quadratic equation for $\gamma_1 = \gamma_{1r} + i\gamma_{1i}$ with the solution

$$\gamma_{1r}^2 = -\widetilde{R}_x = -R_x\cos^2\theta + \langle r\rangle \sin^2\theta,$$
$$R_x = \frac{\langle\beta_1\rangle - c\langle\beta_2\rangle}{\langle\alpha_2\rangle} - \frac{(\langle\beta_2\rangle - c\langle\alpha_2\rangle + \langle\alpha_1\rangle)^2}{4\langle\alpha_2\rangle^2}. \tag{10}$$

Here the angular brackets denote the average $\left(\langle\alpha_2\rangle = \frac{1}{\lambda}\int_0^\lambda \alpha_2(\xi)\,d\xi\right)$ and λ is the wavelength of the wave solution (q_0, h_0). If $\tilde{R}_x < 0$ then it follows from (10) that the solution (q_0, h_0) is unstable to long-wavelength perturbations. If $\tilde{R}_x > 0$, then the quantity γ_1 is purely imaginary and then it is necessary to consider the next approximation in ε.

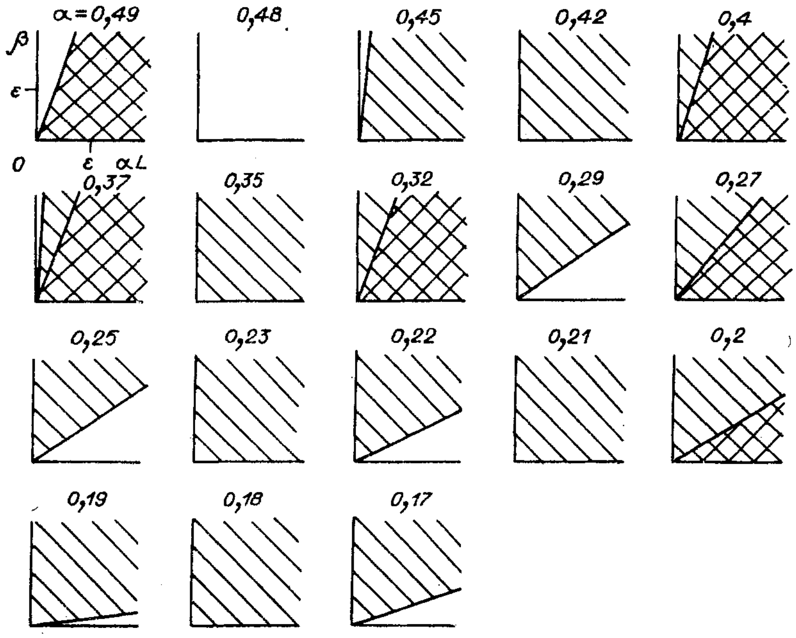

Fig. 5

From the existence condition for a solution we obtain for $\gamma_2 = \gamma_{2r} + i\gamma_{2i}$ a rather complicated linear equation. Since (10) has two solutions, there are the two values $\gamma_{2r}^{1,2}(\theta)$. The wave solution (q_0, h_0) is stable if both of the values of γ_{2r} are positive.

In Fig. 5 the regions of amplified long-wavelength perturbations are shaded for different waves of the second family (the wave numbers are given in Fig. 5) and for p = 5. The regions where the instability increment ~ε ($\tilde{R}_x < 0$) are shown by cross hatching. The wave solution with α = 0.48 is stable to perturbations with small β and L, but additional calculations show that it is unstable to perturbations with large β and L (as in the case of waves of the first family with α = 0.79 in Fig. 1).

It follows from Fig. 5 and also calculations with other values of p that practically all wave solutions of the second family are unstable to long-wavelength three-dimensional perturbations, although there exist many bands of stability to plane perturbations (with β = 0).

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